Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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6.6 Rayleigh-Darcy (or Horton-Rogers-Lapwood) instability in a porous layer

6-6-Lapwood.tex

Nield & Bejan, Chapter 6 **Convection in Porous Media**
Related: Rayleigh-Bernard Problem (Chandrsekhar, Chapter II, **Hydrodynamic and Hydromagnetic Stability**)

If a layer of viscous fluid is heated from below, instability can occur and leads to convection cells important in meteorology. (Rayleigh-Benard Problem).
If a saturated porous layer is heated from below, similar instabilily and convection can occur. This is of basic interest to geothermal convection and is relevant to the complex problem of heat transport due to the burial of nuclear waste in mountains or in a seabed.

To give some visual ideas of what can happen in porous media, we shall borrow some photographic evidence for the mathematically similar Rayleigh-Benard problem of a pure fluid layer heated form below.

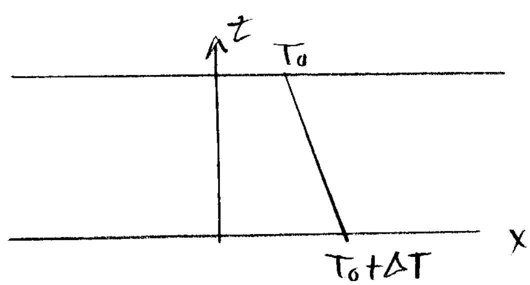

Figure 6.6.1: A saturated porus medium in geothermal gradient

Static state

$$\mathbf{u} = 0, \quad \frac{\partial}{\partial t} = 0$$

Hence

$$T_S = T_o + \Delta T \left(1 - \frac{z}{h}\right) \tag{6.6.1}$$

Static equilibrium:

$$\frac{dp_S}{dz} = -\rho_o g \left[T_o - \beta \Delta T \left(1 - \frac{z}{h}\right)\right]$$

hence

$$p_S = p_o - \rho_o g \left\{ T_o z + \frac{\beta \Delta T}{2} \left(\frac{z^2}{h} - 2z \right) \right\} \tag{6.6.2}$$

Consider the perturbed state of small disturbances:

$$\mathbf{u} = 0 + \mathbf{u}', \quad T = T_S + T', \quad P = p_S + p' \tag{6.6.3}$$

then

$$\nabla \cdot \mathbf{u}' = 0 \tag{6.6.4}$$

$$0 = -\nabla p' - \frac{\mu}{k} \mathbf{u}' + \beta g \rho_o T' \mathbf{k} \tag{6.6.5}$$

$$(\rho C)_m \frac{\partial T'}{\partial t} + (\rho C)_f \mathbf{u}' \cdot \nabla T_S = K_m \nabla^2 T' \tag{6.6.6}$$

6.6.1 Non-dimensionalization

. Let $K_m, \kappa_m = K_m/(\rho C)_f$ be the conductivity and diffusivity of the mixture, and k the permeability. Define

$$(x, y, z) \to h(x^*, y^*, z^*), \; t \to \frac{\sigma h^2}{\kappa_m} t^*,$$
$$\mathbf{u}' \to \frac{\kappa_m}{h} \mathbf{u}^*, \; T' = \Delta T \theta, \; p' \to \frac{\mu \kappa_m}{k} p^* \tag{6.6.7}$$

Then, after omitting * for brevity, we get

$$\nabla \cdot \mathbf{u} = 0 \tag{6.6.8}$$

$$0 = -\nabla p - \mathbf{u} + Ra\, \theta \mathbf{k} \tag{6.6.9}$$

$$\boxed{\frac{\partial \theta}{\partial t} - w = \nabla^2 \theta} \tag{6.6.10}$$

where

$$Ra = \frac{\rho_f g k \beta \Delta T h}{\mu \kappa_m} \quad \text{Rayleigh number in a porous medium} \tag{6.6.11}$$

is the Rayleigh number (ratio of buoyancy force to diffusive resistence) of the porous medium. In Benard's problem, Rayleigh number is defined as

$$Ra = \frac{\rho g \Delta T h^4}{\mu \kappa}, \quad \text{Rayleigh number in a pure fluid} \tag{6.6.12}$$

Sometimes one calls

$$D = \frac{k}{h^2}, \text{Darcy number} \tag{6.6.13}$$

Darcy number so that Rayleigh number of a porous medium is the product of the traditional Rayleigh number and Darcy number.

Taking the curl of (6.6.9),

$$\nabla \times \mathbf{u} = Ra(\mathbf{i}\theta_y - \mathbf{j}\theta_x) \tag{6.6.14}$$

Note that the z compoent of the vorticity vector is zero.

Taking the curl again and using

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$$

we get

$$\nabla^2 \mathbf{u} = -Ra\left[\mathbf{i}\theta_{xz} + \mathbf{j}\theta_{yz} - \mathbf{k}(\theta_{xx} + \theta_{yy})\right]$$

Taking the z component, we get

$$\boxed{\nabla^2 w = Ra \nabla_H^2 \theta} \tag{6.6.15}$$

where

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6.6.16}$$

is the horizontal Laplacian.

Equations (6.6.10) and (6.6.15) couple the two unknowns w and θ. The boundary conditions are

$$w = \theta = 0, \quad z = 0, 1 \tag{6.6.17}$$

After they are solved the other velocity components and pressure can be found.

6.6.2 Solution for sinusoidal disturbances

Let

$$(w, \theta) = (W(z), \Theta(z)) \exp(ilx + imy - i\omega t) \tag{6.6.18}$$

and

$$D = \frac{d}{dz}$$

then from (6.6.10),

$$-i\omega\Theta - W = (D^2 - a^2)\Theta \tag{6.6.19}$$

and from (6.6.15)

$$(D^2 - a^2)W = -a^2 Ra \Theta \tag{6.6.20}$$

where

$$a^2 = \ell^2 + m^2 \tag{6.6.21}$$

The boundary conditions are :

$$W = \Theta = 0, \quad z = 0, 1 \tag{6.6.22}$$

Because the equations and the boundary conditons are homogeneous, the problem for W and θ is an eigenvalue problem.

Note that l, m, a are related to dimensional wave numbers by

$$l = k_x h = \frac{2\pi h}{L_x}, \quad m = k_y h = \frac{2\pi h}{L_y}, a = kh = 2\pi h \left(\frac{1}{L_x^2} + \frac{1}{L_y^2} \right)^{1/2} \quad \text{with} \quad k = \sqrt{k_x^2 + k_y^2} \tag{6.6.23}$$

6.6.3 Principle of exchange of stabilities

We shall first show that ω must be purely imaginary. Multiplying (6.6.20) by W^* and integrating from $z = 0$ to $z = 1$, we get, after partial integration and using the boundary conditions,

$$-\int_0^1 (|DW|^2 + a^2|W|^2)dz = -a^2 Ra \int_0^1 W^* \Theta dz \tag{6.6.24}$$

Similarly we multipy (6.6.19) by Θ^* and integrating from $z = 0$ to $z = 1$, and get

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = -i\omega \int |\Theta|^2 dz - \int_0^1 W\Theta^* dz \tag{6.6.25}$$

Taking the complex conjugate of the second equation

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = i\omega^* \int |\Theta|^2 dz - \int_0^1 W^*\Theta dz \tag{6.6.26}$$

Eqs (6.6.24) and (6.6.26) can be combined by eliminating the cross product terms, ,

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = -i\omega^* \int |\Theta|^2 dz - \frac{1}{a^2 Ra} \int_0^1 (|DW|^2 + a^2|W|^2)dz \tag{6.6.27}$$

Since all integrals above are real, $-i\omega* = -\omega_i - i\omega_r$ must also be real. We conclude that

$$\omega_r = 0, \quad \text{hence} \quad -i\omega = -\omega_i \tag{6.6.28}$$

Marginal stability (the threshold of instability) occurs at $\omega_r = \omega_i = 0$. If $\omega_i > 0$, the static state is unstable; $\omega_i = 0$, marginally stable; if $\omega_i < 0$, stable. A problem where the eigenfrequency is real so that marginal instabililty occurs when $\omega = 0$ is said to obey the *principle of exchange of stabilities.*

6.6.4 Solution to eigenvalue problem

Consider the situation at marginal stability : $\omega = 0$,

$$(D^2 - a^2)W = -a^2 Ra\Theta \tag{6.6.29}$$

$$-W = (D^2 - a^2)\Theta \tag{6.6.30}$$

Eliminating Θ, we get

$$(D^2 - a^2)^2 W = a^2 Ra W \tag{6.6.31}$$

subject to

$$W = 0, \quad D^2 W = 0, \quad z = 0, 1 \tag{6.6.32}$$

Expanding (6.6.31)

$$D^4 W - 2a^2 D^2 W + a^4 W = a^2 Ra W \tag{6.6.33}$$

Clearly $D^4 W = 0$ on $z = 0, 1$. Differentiating (6.6.33) twice we see that $D^6 = 0$ on $z = 0, 1$. Repeating the process we find

$$D^{(2m)} = 0, \quad m = 1, 2, 3, \cdots, \quad \text{on} \quad z = 0, 1 \tag{6.6.34}$$

Therefore the eigensolution must be

$$W \sim \sin j\pi z \tag{6.6.35}$$

To satisfy (6.6.31) it is necessary that

$$Ra = \frac{[j^2\pi^2 + a^2]^2}{a^2}, \quad \text{for } j = 1, 2, 3..., \tag{6.6.36}$$

which is the eigenvalue condition. For any j, Ra becomes unbounded for both $a^2 \to 0$ and $a^2 \to \infty$ and is curve concave upward in the plane of a^2(abscissa) vs, Ra (ordinate).

The lowest threshold occurs at $j = 1$, and

$$\frac{\partial Ra}{\partial a^2} = 0$$

i.e.,

$$a^2 = \pi^2 \tag{6.6.37}$$

or

$$Ra_c = 4\pi^2 = 39.48 \tag{6.6.38}$$

6.6.5 Possible convection patterns

This is similar to Benard's problem which has been exhaustively studied theoretically and experimentally. There are many possibilities. Let us consider the lowest mode only with $j = 1$.

2-Dimensinal Rolls : $(\ell = \pi, m = 0)$

Take

$$w = \cos \pi x \sin \pi z \tag{6.6.39}$$

then from mass conservation,

$$u = -\sin \pi x \cos \pi z \tag{6.6.40}$$

The dimensionless wavelength is $L_x = L_y = 2$. Along lines $x = 0, \pm n, n = 1, 2, 3, 4, ..., u = 0$ but $w \neq 0$. Along $x = 0, \pm 2m$, $w > 0$, hence fluid rises vertically. Along $x = \pm 2m - 1$, $w < 0$ hence fluids sinks vertically. Along $z = 0$ and $1, w = 0$. On the bottom ($z = 0$), $u > 0$ while on the top ($z = 1$), if $0 < x < 1, 2 < x < 3, 4 < x < 5, ...$. The streamlines are shown in Figure 6.6.2.

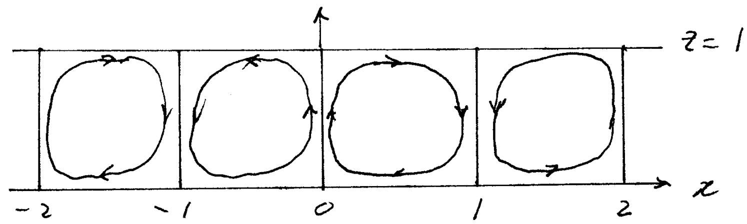

Figure 6.6.2: Rolls in a period

Rectangular cells: ($\ell = m = \pi/\sqrt{2}$).

$$w = \cos\frac{\pi x}{\sqrt{2}}\cos\frac{\pi y}{\sqrt{2}}\sin\pi z \tag{6.6.41}$$

From the z component of the vorticity equation (6.6.14),

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{6.6.42}$$

and from continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \tag{6.6.43}$$

By cross differentiation, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 w}{\partial z \partial y} = \frac{\pi^2}{2\sqrt{2}} \sin\frac{\pi x}{\sqrt{2}} \cos\frac{\pi y}{\sqrt{2}} \cos\pi z \tag{6.6.44}$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 w}{\partial z \partial x} = \frac{\pi^2}{2\sqrt{2}} \cos\frac{\pi x}{\sqrt{2}} \sin\frac{\pi y}{\sqrt{2}} \cos\pi z \tag{6.6.45}$$

These are easily solved to give

$$u = -\frac{1}{\sqrt{2}} \sin\frac{\pi x}{\sqrt{2}} \cos\frac{\pi y}{\sqrt{2}} \cos\pi z \tag{6.6.46}$$

and

$$v = -\frac{1}{\sqrt{2}} \cos\frac{\pi x}{\sqrt{2}} \sin\frac{\pi y}{\sqrt{2}} \cos\pi z \tag{6.6.47}$$

The streamlines in a horizontal plane is shown in Figure 6.6.4.

<u>Hexagonal cells</u>: See Chandrasekhar.

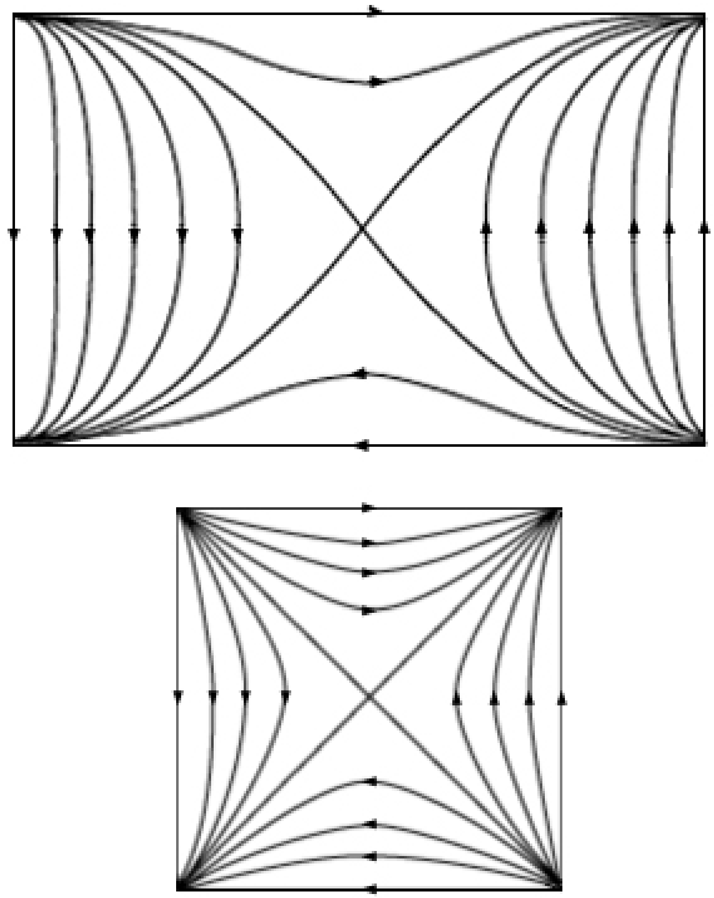

Figure 6.6.4: *Rectangular and square cells in a pure fluid heated from below.*
(Adapted from Chandrasekhar, Hydrodynamic and Hydromagnetic Stability).